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Bases for Analytic Functions on Infinitely Connected Compact Sets

VICTOR MANJARREZ*

Department of Mathematics, California State College, Hayward, California 94542

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1. INTRODUCTION

Let K be a compact subset of the plane. K need not be connected nor of finite connectivity, but it will be assumed that each component of the complement of K has a Green's function. Work of J. L. Walsh is utilized to construct an interpolation basis $(Q_n : n = 0, 1, 2,...)$, of rational functions with poles off K, for the functions analytic on K, with the topology of uniform convergence on compact subsets of some open neighborhood of K. A sequence $(Q_n : n = -1, -2, -3,...)$ of rational functions with poles on K is then constructed such that $\sum_{n=0}^{\infty} Q_n(z)\, Q_{-n-1}(w) = 1/(w - z)$ for w and z on suitable subsets of the plane. This is used to construct, for any basis $(P_k : k = 0, 1, 2,...)$ for the functions analytic on K, a sequence $(P_k : k = -1, -2, -3,...)$ such that $\sum_{k=0}^{\infty} P_k(z)\, P_{-k-1}(w) = 1/(w - z)$. This construction is applied to the expansion of topological vector space-valued functions holomorphic on K. Examples are given involving orthonormal polynomials and Faber polynomials.

2. INTERPOLATION BASES

Let K be a compact subset of the plane, each component of whose complement having a Green's function. Such a set K will be called a *coregular* set. Let e be the unbounded component of the complement of K, let F be its Green's function with pole at infinity, and let K_0 be the complement of e. For each extended real number $x > 1$, let $E(x)$ be the set of all z such that

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either z is in K_0 or $\exp(F(z)) < x$; let $L(x)$ be the set of all finite z such that $\exp(F(z)) = x$ (oriented so that $E(x)$ is to the left of $L(x)$), and let $e(x)$ be the set of all z such that either z is the point at infinity or $\exp(F(z)) > x$. Choose points $(b(n)\colon n = 1, 2, 3,...)$ in K such that

$$| \lim(z - b(1))(z - b(2)) \cdots (z - b(n))|^{1/n} = D \exp(F(z)), \tag{2.1}$$

where D is the capacity or transfinite diameter of K_0, and the convergence is uniform on compact subsets of e. (We may, for example, choose for $k = 1, 2,...$ points $b(k, 1), b(k, 2),..., b(k, k)$ so as to maximize the modulus of the k-th Vandermonde determinant $\prod_{i<j}^{k} (z(i) - z(j))$ on K_0 and let $(b(n))$ be the sequence $b(1, 1), b(2, 1), b(2, 2), b(3, 1),..., b(k, k),...$ [6, pp. 170, 157]. The maximum principle insures that these points lie on the boundary of K_0 and thus in K.) Uniform convergence on compact subsets of an open set O containing a set S will be called *compact-open* convergence in O or compact-open convergence on S.

Let

$$Q_0(z) = 1; \qquad Q_n(z) = (z - b(1)) \cdots (z - b(n)),$$
$$Q_{-n}(z) = 1/Q_n(z), \qquad n = 1, 2, \dots . \tag{2.2}$$

If x is any number > 1, then by (2.1), $\sum_{n=0} Q_n(z)\, Q_{-n-1}(w)$ converges uniformly for z on any compact subset of $E(x)$ and w on any compact subset of $e(x)$. If f is any function analytic on K_0, then f is analytic in $E(x)$ for some $x > 1$, so

$$f(z) = \sum_{n=0}^{\infty} B(n)\, Q_n(z), \qquad B(n) = (1/2\pi i) \int_{L(y)} f(t)\, Q_{-n-1}(t)\, dt, \tag{2.3}$$

for any y such that $1 < y < x$, the convergence of the series being compact-open in $E(x)$ [6, pp. 65, 76]. The coefficients in this compact-open representation of f are uniquely determined since, for every $n, k = 0, 1, 2,...,$

$$(1/2\pi i) \int_L Q_k(t)\, Q_{-n-1}(t)\, dt = \delta_{nk},$$

where $\delta_{nk} = 1$ if $n = k$, and $= 0$ otherwise. Now if w is any finite point in e, then w is in $e(x)$ for some $x > 1$, and if z is in $E(x)$, then, from (2.3),

$$1/(w - z) = \sum_{n=0}^{\infty} Q_n(z)(1/2\pi i) \int_{L(x)} (Q_{-n-1}(t)/(w - t))\, dt = \sum_{n=0}^{\infty} Q_n(z)\, Q_{-n-1}(w),$$

the convergence being uniform for z on any compact subset of $E(x)$ and w on any compact subset of $e(x)$.

If f is analytic in $e(x) \cap E(y)$ where $1 < x < y$, and if z is any point in $e(x) \cap E(y)$, choose r, s such that $x < r < \exp(F(z)) < s < y$. Then

$$\begin{aligned} f(z) &= (1/2\pi i) \int_{L(s)} (f(t)/(t-z))\, dt - (1/2\pi i) \int_{L(r)} (f(t)/(t-z))\, dt \\ &= \sum_{n=0}^{\infty} (Q_n(z)(1/2\pi i) \int_{L(s)} f(t)\, Q_{-n-1}(t)\, dt \\ &\quad + Q_{-n-1}(z)(1/2\pi i) \int_{L(r)} f(t)\, Q_n(t)\, dt) \end{aligned}$$

and so

$$f(z) = \sum_{n=-\infty}^{\infty} B(n)\, Q_n(z), \qquad B(n) = (1/2\pi i) \int_L f(t)\, Q_{-n-1}(t)\, dt;$$

where $L =$ any $L(s)$ between z and $L(y)$ for nonnegative n; $L =$ any $L(r)$ between z and $L(x)$ for negative n; and the convergence of the series representing f is compact-open in $e(x) \cap E(y)$.

Now let e_j be any fixed bounded component of the complement of K, let $a(= a(j))$ be a fixed point in e_j, let $e = H(e_j)$ where H is the linear fractional transformation given by $H(z) = 1/(z-a)$, and let K_0 be the complement of e. If F is Green's function for e with pole at infinity, then by uniqueness, $F_j(z) = F(H(z))$ is Green's function for e_j with pole at a. If we define $E(x)$, $L(x)$ and $e(x)$ as above (using $F(z)$), then we can define $E_j(x)$, $L_j(x)$ and $e_j(x)$ as the images of $E(x)$, $L(x)$ and $e(x)$ under H^{-1} (or equivalently, we can define them directly in terms of F_j) with $L_j(x)$ oriented so that $E_j(x)$ is to its left. Let $(b(n)$: $n = 1, 2, 3,...)$ belong to the boundary of K_0 (= outer boundary of $H(K)$) and be such that (2.1) is satisfied, and let $b(j, n) = H^{-1}(b(n))$, $n = 1, 2, 3,\ldots$. If f is analytic in $E_j(x)$ for some $x > 1$, then $g(u) = f(H^{-1}(u))$ is analytic in $E(x)$; so, from (2.3), $f(z) = g(u) = \sum_{n=0}^{\infty} B(n)\, Q_n(u)$ where $Q_0(u) = 1$;

$$\begin{aligned} Q_n(u) &= (u - b(1)) \cdots (u - b(n)) \\ &= \frac{(z - b(j,1)) \cdots (z - b(j,n))}{(z-a)^n\, (a - b(j,1)) \cdots (a - b(j,n))} = 1/Q_{-n}(u), \end{aligned}$$

for $n = 1, 2, 3,...$; while

$$\begin{aligned} B(0) &= (1/2\pi i) \int_{L(y)} Q_{-1}(v)\, g(v)\, dv \\ &= ((b(j,1) - a)/2\pi i) \int_{L_j(y)} \frac{f(t)\, dt}{(t - b(j,1))(t-a)} \\ &= f(b(j,1)) = (1/2\pi i) \int_{L_j(y)} (f(t)/(t - b(j,1)))\, dt + f(\infty) \end{aligned}$$

for any y such that $1 < y < x$, and

$$\begin{aligned} B(n) &= (1/2\pi i) \int_{L(y)} Q_{-n-1}(v)\, g(v)\, dv \\ &= ((b(j, n+1) - a)/2\pi i) \\ &\quad \times \int_{L_j(y)} \frac{(t-a)^{n-1} (a - b(j,1)) \cdots (a - b(j,n)) f(t)\, dt}{(t - b(j,1)) \cdots (t - b(j, n+1)} \end{aligned}$$

for $n = 1, 2, 3, \ldots$. Thus if we define

$$\begin{aligned} Q_{j0}(z) &= 1; \\ Q_{jn}(z) &= \frac{(z - b(j,1)) \cdots (z - b(j,n))}{(z-a)^n (b(j,1) - a) \cdots (b(j,n) - a)}, \qquad n = 1, 2, 3, \ldots; \\ Q_{j,-1}(z) &= \frac{1}{z - b(j,1)}; \\ Q_{j,-n}(z) &= \frac{(z-a)^{n-2} (b(j,1) - a) \cdots (b(j,n) - a)}{(z - b(j,1)) \cdots (z - b(j,n))}, \qquad n = 2, \ldots; \end{aligned} \tag{2.4}$$

then $f(z) - f(\infty) = \sum_{n=0}^{\infty} B(j, n)\, Q_{jn}(z)$, where

$$B(j, n) = (1/2\pi i) \int_{L_j(y)} f(t)\, Q_{j,-n-1}(t)\, dt$$

for any y such that $1 < y < x$, and the convergence of the series is compact-open in $E_j(x)$. The coefficients in this compact-open representation of f are uniquely determined, since $B(j, 0) = f(b(j, 1)) - f(\infty)$ while

$$(1/2\pi i) \int_{L_j} Q_{jk}(t)\, Q_{j,-n-1}(t)\, dt = \delta_{nk}, \qquad n = 1, 2, 3, \ldots; \qquad k = 0, 1, 2, \ldots. \tag{2.5}$$

Moreover, as above, if w is any point different from a in e_j, then w is in $e_j(x)$ for some $x > 1$, and if z is in $E_j(x)$, then

$$\begin{aligned} 1/(w - z) &= \sum_{n=0}^{\infty} Q_{jn}(z)(1/2\pi i) \int_{L_j(x)} (Q_{j,-n-1}(t)/(w - t))\, dt \\ &= \sum_{n=0}^{\infty} Q_{jn}(z)\, Q_{j,-n-1}(w), \end{aligned} \tag{2.6}$$

where the convergence is uniform for z on any compact subset of $E_j(x)$ and

w on any compact subset of $e_j(x)$. Again, as above, if f is analytic in $e_j(x) \cap E_j(y)$, then

$$f(z) = \sum_{n=-\infty}^{\infty} B(j, n)\, Q_{jn}(z), \qquad B(j, n) = (1/2\pi i) \int_{L_j} f(t)\, Q_{j,-n-1}(t)\, dt \tag{2.7}$$

where $L_j =$ any $L_j(s)$ between z and $L_j(y)$ for nonnegative n; $L =$ any $L_j(r)$ between z and $L_j(x)$ for negative n; and the convergence is compact-open in $e_j(x) \cap E_j(y)$.

Finally, let K be any coregular set, let $(e_j : j = 0, 1, 2,...)$ be the components of the complement of K, with e_0 the unbounded one; choose a point a_j in each e_j, with a_0 the point at infinity; and let F_j be Green's function for e_j with pole at a_j. Let X be the set of all sequences $(x(0), x(1), x(2)...)$ such that each $x(j)$ is an extended real number > 1 and all but finitely many of the $x(j) = \infty$. If x is in X, define $E_j(x) = E_j(x(j))$, $e_j(x) = e_j(x(j))$ and $L_j(x) = L_j(x(j))$ as above, let $E(x)$ be the intersection of the $E_j(x)$, let $e(x)$ be the union of the $e_j(x)$, and let $L(x)$ be the union of the $L_j(x)$, where all of these unions and intersections are taken over those j for which $x(j) < \infty$. For each $j = 0, 1, 2,...$ choose $(b(j, n)$: $n = 1, 2, 3,...)$ in K (on the boundary of e_j) as above, and define $(Q_{jn} : n = 0, \pm 1, \pm 2,...)$ by (2.2) when $j = 0$, taking $b(j, n) = b(n)$; and by (2.4) when $j > 0$, taking $a(j) = a$; for negative n we make the convention that Q_{jn} is defined as just stated on e_j, and as identically zero on the complement of e_j. The sequence

$$Q = (Q_{jn} : j = 0, 1, 2,...;\ n = 1, 2, 3,...) \cup (Q_{00})$$

will be called an *interpolation basis* for K. We sum up in

THEOREM 1. *Let K be a coregular set and let Q be an interpolation basis for K. Then for any x in X, (2.6) holds for each $j = 0, 1, 2,...,$ the convergence being uniform for z on any compact subset of $E_j(x)$ and w on any compact subset of $e_j(x)$. If f is any function analytic on K, then there exists x in X such that f is analytic in $E(x)$, and f can be expanded uniquely in the form*

$$f(z) = B(0, 0) + \sum_{j=0}^{\infty} \sum_{n=1}^{\infty} B(j, n)\, Q_{jn}(z),$$

$$B(0, 0) = \sum_{j=0}^{\infty} (1/2\pi i) \int_{L(y)} f(t)\, Q_{j,-1}(t)\, dt, \tag{2.8}$$

$$B(j, n) = (1/2\pi i) \int_{L(y)} f(t)\, Q_{j,-n-1}(t)\, dt, \qquad j = 0, 1, 2,...; \qquad n = 1, 2,...,$$

for any $y < x$ in X (with $y(0) < \infty$), the convergence of the series being compact-open in $E(x)$. Moreover, for $j = 0, 1, 2,...,$ (2.5) holds for $L_j =$ any $L_j(x)$

such that $x(j) < \infty$. *Finally, if* f *is analytic in* $e(x) \cap E(y)$ *for some* $x < y$ *in* X, *then* (2.7) *holds for* z *in* $e_j(x) \cap E_j(y)$, *with compact-open convergence in* $e_j(x) \cap E_j(y)$.

Proof. If f is analytic on K, it is analytic in some neighborhood O of K, and there exists a positive integer N such that e_j is contained in O for all $j > N$. Thus there exists x in X with $x(j) = \infty$ for $j > N$, such that f is analytic in $E(x)$. If z is in $E(x)$, then z is in $E(y)$ for some $y < x$ in X; so, from (2.6) (we may assume $y(0) < \infty$),

$$f(z) = (1/2\pi i) \int_{L(y)} (f(t)/(t - z))\, dt = \sum_{j=0}^{N} (1/2\pi i) \int_{L_j(y)} (f(t)/(t - z))\, dt$$

$$= \sum_{j=0}^{N} \sum_{n=0}^{\infty} Q_{jn}(z)(1/2\pi i) \int_{L_j(y)} f(t)\, Q_{j,-n-1}(t)\, dt,$$

and thus (2.8) holds, by the convention that $Q_{j,-n-1}(t) = 0$ for t on $L_k(y)$, $k \neq j$, and the fact that

$$\int_{L_j(y)} f(t)\, Q_{j,-n-1}(t)\, dt = 0, \qquad j > N.$$

The uniqueness of the coefficients of the nonconstant terms in this compact-open representation of f follows from (2.5) and the fact that

$$\int_{L_j(y)} Q_{mk}(t)\, Q_{j,-n-1}(t)\, dt = 0$$

for $j, m, k = 0, 1, 2,\ldots, n = 1, 2, 3,\ldots$, and $m \neq j$; since in each case, except $j = 0$, the integrand is analytic on the bounded set $e_j(x)$; and when $j = 0$, $L_j(y)$ is outside all the poles of the integrand, and the degree of the denominator in the integrand is larger by two than the degree of the numerator. The uniqueness of the coefficient of the constant term follows from the uniqueness of the other terms. This completes the proof of the theorem.

3. Bases

Let K be a coregular set, let x be in X and let $P = (P_k : k = 0, 1, 2,\ldots)$ be a sequence of functions analytic in $E(x)$ (except possibly at the points $a(j)$). If every function f analytic in $E(x)$ can be expanded uniquely in the form

$$f(z) = \sum_{k=0}^{\infty} A(k)\, P_k(z), \tag{3.1}$$

the convergence being compact-open in $E(x)$, then P is called a *basis* for $E(x)$ (with compact-open convergence). If each P_k is analytic on a fixed open set containing $\mathrm{Cl}(E(x))$ = closure of $E(x)$; and if every function f analytic on $\mathrm{Cl}(E(x))$ can be expanded uniquely in the form (3.1), the convergence being compact-open on $\mathrm{Cl}(E(x))$, then P will be called a basis for $\mathrm{Cl}(E(x))$. If P is a basis for $E(x)$ and $\mathrm{Cl}(E(x))$ for every x in X, then P will be called a basis for K. Every interpolation basis for K is a basis for K.

THEOREM 2. *Let P be a basis for $\mathrm{Cl}(E(x))$ and $E(y)$, for some $x < y$ in X. Then there exists a sequence $(P_k : k = -1, -2, -3,...)$ of functions analytic on $e(x)$ such that, for every q in X with $x < q < y$, there exist r, s in X with $x < r < q < s < y$ and*

$$1/(w - z) = \sum_{k=0}^{\infty} P_k(z)\, P_{-k-1}(w), \tag{3.2}$$

the convergence being uniform for z on $\mathrm{Cl}(E(y))$ and w on $\mathrm{Cl}(e(s))$, and also uniform for z on $\mathrm{Cl}(E(r))$ and w on $\mathrm{Cl}(e(q))$.

Proof. Let Q be an interpolation basis for K. If $x(j) < \infty$, then Q_{jn} is analytic on $E(x)$ for $n = 0, 1, 2,...$, and so there exist complex numbers $G(j, n, k)$ such that

$$Q_{nj}(z) = \sum_{k=0}^{\infty} G(j, n, k)\, P_k(z), \tag{3.3}$$

the convergence being compact-open on $E(x)$. If $(B(n): n = 0, 1, 2,...)$ is any sequence of complex numbers such that $f(z) = \sum_{n=0}^{\infty} B(n)\, Q_{jn}(z)$ converges compact-openly on $\mathrm{Cl}(E(x))$, then f can be expanded in the form (3.1) with $A(k) = \sum_{n=0}^{\infty} B(n)\, G(j, n, k)$ [5, Theorem 9.3, p. 444]. From (2.6) it follows that

$$P_{j,-k}(z) = \sum_{n=0}^{\infty} G(j, n, k-1)\, Q_{j,-n-1}(z), \qquad k = 1, 2, 3,..., \tag{3.4}$$

converge compact-openly on $e_j(x)$. For $k = 1, 2, 3,...$ we define $P_{-k}(z) = P_{j,-k}(z)$ for z in e_j, for those j for which $x(j) < \infty$. Since P is a basis for $E(y)$, there exists an s in X, with $q < s < y$, such that, if

$$N_{jnm}(z, w) = \left| \sum_{k=0}^{m} G(j, n, k)\, P_k(z)\, Q_{j,-n-1}(w) \right|,$$

then

$$\sum_{n=0}^{\infty} \max(\max(N_{jnm}(z, w) : z \text{ in } \mathrm{Cl}(E(q)), w \text{ in } \mathrm{Cl}(e(s))) : m = 0, 1, 2,...) < \infty$$

for each j satisfying $q(j) < \infty$ [5, Theorem 7.3, Corollary 2, p. 442]. Thus reversal in the order of summation is justified in the following sum, based on (3.4), for z in $\mathrm{Cl}(E(q))$ and w in $\mathrm{Cl}(e_j(s))$:

$$\begin{aligned}\sum_{k=0}^{\infty} P_k(z)\, P_{-k-1}(w) &= \sum_{k=0}^{\infty} P_k(z)\, P_{j,-k-1}(w) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} G(j, n, k)\, P_k(z)\, Q_{j,-n-1}(w) \\ &= \sum_{n=0}^{\infty} Q_{nj}(z)\, Q_{j,-n-1}(w) = 1/(w - z)\end{aligned}$$

by (3.3) and Theorem 1. A similar argument, which makes use of [5, Theorem 7.3, p. 441] completes the proof of the theorem.

An examination of the proof of Theorem 2 yields the following

COROLLARY. *If P is a basis for K, then there exists a sequence $(P_k : k = -1, -2, -3,...)$ of functions analytic on the complement of K with the property that, for every q in X, there exist r, s in X, with $r < q < s$ and $s(j) < \infty$ whenever $q(j) < \infty$, such that* (3.2) *holds, the convergence being uniform for z in* $\mathrm{Cl}(E(q))$ *and w in* $\mathrm{Cl}(e(s))$, *and also uniform for z in* $\mathrm{Cl}(E(r))$ *and w in* $\mathrm{Cl}(e(q))$.

Compare [5, Theorem 24.1, p. 466] for Theorem 2 in the special case in which K is a disk, and [4, Theorems 1, 5] for Theorems 1 and 2 in the special case where the complement of K is connected and bounded by finitely many nonintersecting analytic Jordan curves.

4. VECTOR VALUED FUNCTIONS

Let V be a complex Hausdorff locally convex topological vector space whose topology T has the property that the closed convex balanced hull of every T-compact set is T-compact. V will satisfy this condition, for example, if it is quasi-complete [3, Proposition 7, p. 234]. Thus V can be any complex Banach or Fréchet space. Let O be an open subset of the complex plane, and let f be a function from O into V such that, for every continuous linear functional h on V, the complex function $h(f(z))$ is analytic in O. The function f is said to be *weakly holomorphic* in O. It follows that f is T-continuous, and that if L is any rectifiable Jordan curve which, together with its interior, lies in O, then

$$f(z) = (1/2\pi i) \int_L (f(t)/(t-z))\, dt$$

for all z in the interior of L, where the integral is the T-limit of partial vector sums [2, Theorem 1, p. 37].

THEOREM 3. *Let K be a coregular subset of the complex plane, let P be a basis for K (as in Section 3), and let V be a complex Hausdorff locally convex topological vector space, whose topology T has the property that the closed convex balanced hull of every T-compact set is T-compact. Then every V-valued function f which is weakly holomorphic in an open set containing K can be expanded uniquely in the form (3.1), where each $A(k)$ is in V, and the convergence is T-compact-open on K (that is, (3.1) converges with respect to T, uniformly on compact subsets of some open neighborhood of K).*

Proof. Let y in X be such that f is weakly holomorphic on $E(y)$. If z is in $E(y)$, then z is on $L(q)$ for some $q < y$ in X, and by the corollary to Theorem 2, there exists an s in X with $q < s < y$, and a sequence $(P_k : k = -1, -2, -3, ...)$ of functions analytic on the complement of K, such that (3.2) holds, the convergence being uniform for z on $\mathrm{Cl}(E(q))$ and w on $\mathrm{Cl}(e(s))$. But then $\sum_{k=0}^{\infty} P_k(z)\, P_{-k-1}(t) f(t)$ converges with respect to T, uniformly for z on $\mathrm{Cl}(E(q))$ and t on $L(s)$ (since $f(L(s))$ is T-compact in V), and so (we may assume $s(0) < \infty$)

$$f(z) = (1/2\pi i) \int_{L(s)} (f(t)/(t-z))\, dt = (1/2\pi i) \int_{L(s)} \sum_{k=0}^{\infty} P_k(z)\, P_{-k-1}(t) f(t)\, dt$$

$$= \sum_{k=0}^{\infty} P_k(z)(1/2\pi i) \int_{L(S)} P_{-k-1}(t) f(t)\, dt = \sum_{k=0}^{\infty} A(k)\, P_k(z),$$

where the uniform convergence, for z on $\mathrm{Cl}(E(q))$ and t on $L(s)$, of the series under the integral sign justifies the reversal of the order of integration and summation (since the T-continuity of a V-valued function g on the compact set $L(s)$ implies that the integral of g around $L(s)$ is in the compact set formed by multiplying the closed convex balanced hull of $g(L(s))$ by the length of $L(s)$), and also implies the uniform convergence of the last series for z on $\mathrm{Cl}(E(q))$. Each $A(k)$ is independent of s as long as $q < s$ (and $s(0) < \infty$), and so the series in (3.1) is well-defined and converges uniformly in a neighborhood of each point of $E(x)$, and therefore compact-openly on $E(x)$. If

h is any continuous linear functional on V, then $h(f(z)) = \sum_{k=0}^{\infty} h(A(k))\, P_k(z)$, since for each z the series in (3.1) converges with respect to T. Thus the numbers $(h(A(k)): k = 0, 1, 2,...)$ are the uniquely determined coefficients in the expansion of the complex function $h(f(z))$ with respect to the basis P, so if $f(z) = \sum_{k=0}^{\infty} B(k)\, P_k(z)$ with T-compact-open convergence, then $h(B(k)) = h(A(k))$ for every T-continuous linear functional h, and so $B(k) = A(k)$. Thus the expansion in (3.1) is unique, and this completes the proof of the theorem.

5. Example

Let K be a coregular set, let e_j be a bounded component of the complement of K, let a $(= a(j))$ be a point in e_j, and let H $(= H_j)$ be the linear fractional transformation given by $H(z) = 1/(z - a)$. Let $e = H(e_j)$ and let K_0 be the complement of e. Let $p = (p_k : k = 0, 1, 2,...)$ be a sequence of polynomials with exact degree of $p_k = k$, which is orthonormal with respect to integration over K_0, or around the boundary of K_0 if it is rectifiable, with any positive continuous function allowed as weight function. Then p is a basis for K_0 [6, pp. 91–7, 125–8]. Similarly, if the boundary of K_0 is an analytic Jordan curve, we may take p to be the sequence of Faber polynomials associated with the curve [1, pp. 86–7]. In any of these cases, any function f analytic on K_0 can be expanded uniquely in the form (3.1), the convergence being compact-open on K_0. If f is analytic on the complement K_j of e_j, then $g(u) = f(H^{-1}(u))$ is analytic on K_0; so $f(z) = g(u) = \sum_{k=0}^{\infty} A(k)\, p_k(u) = \sum_{k=0}^{\infty} A(k)\, p_k(H(z)) = \sum_{k=0}^{\infty} A(k)\, P_k(z)$ where each P_k is a rational function of degree k with pole at a, the convergence being compact-open on K_j, and the coefficients uniquely determined. We define $P_{jk}(z) = P_k(z)$, $k = 0, 1, 2,...,$ and repeat this construction for each $j = 1, 2, 3,...$. Since $(P_{jk} : k = 0\ 1, 2,...)$ is a basis for K_j, Theorems 2 and 3 imply that there exists a sequence $(P_{jk} : k = -1, -2, -3,...)$ of functions analytic on e_j such that any (vector-valued) analytic function f_j on K_j can be expanded compact-openly on K_j in the form

$$f_j(z) - f_j(\infty) = \sum_{k=0}^{\infty} A(k)\, P_{jk}(z), \quad A(k) = (1/2\pi i) \int_{L_j} P_{j,-k-1}(t) f(t)\, dt \qquad (5.1)$$

where $L_j = L_j(x)$ for a suitable x in X. For the case of the unbounded component e_0 of the complement of K we let $P_{0k}(z) = P_k(z)$, and use Theorems 2 and 3 to get the sequence $(P_{0k} : k = -1, -2, -3,...)$ such that any f_0 analytic on the complement K_0 of the unbounded component of the complement of K can be expanded in the form $f_0(z) = \sum_{k=0}^{\infty} A(k)\, P_{0k}(z)$, with the $A(k)$ given as in (5.1). We define $P_{jk}(z) = 0$ for z on K_j, $k = -1, -2, -3,..., j = 0, 1, 2,...$.

Now if f is analytic on K, then (2.8) of Theorem 1 implies that f is the sum of (finitely many) functions f_j, each analytic on K_j, since if $\sum_{n=0}^{\infty} B(j, n)\, Q_{jn}(z)$ converges uniformly on $L_j(x)$, then it converges uniformly on $E_j(x)$. Each f_j can be expanded in the form (5.1) (with the necessary change when $j = 0$), and so f can be expanded compact-openly on K in the form

$$f(z) = A(0, 0) + \sum_{j=0}^{N} \sum_{k=1}^{\infty} A(j, k)\, P_{jk}(z),$$

$$A(0, 0) = \sum_{j=0}^{\infty} (1/2\pi i) \int_{L(y)} f(t)\, P_{j,-1}(t)\, dt,$$

$$A(j, n) = (1/2\pi i) \int_{L(y)} f(t)\, P_{j,-n-1}(t)\, dt, \qquad j = 0, 1, 2,...; \qquad n = 1, 2, 3,...,$$

for a suitable y in X. The uniqueness of this compact-open expansion of f follows exactly as the uniqueness of the expansion in (2.8) since if $j \neq 0$, then $P_{mk}(t)\, P_{j,-n-1}(t)$ is analytic on the bounded set e_j, for m and $k = 0, 1, 2,..., n = 1, 2, 3,..., m \neq j$; if $j = 0$, m and $k = 0, 1, 2,..., n = 1, 2, 3,..., m \neq 0$, then $P_{mk}(t)\, P_{0,-n-1}(t)$ is a rational function with all its poles inside $L_j(y)$ and with denominator of degree at least two greater than the degree of the numerator (from (3.4)); and finally, if $m = j$, then from (5.1) and the fact that $(P_{jk} : k = 0, 1, 2,...)$ is a basis for K_j, we have that

$$(1/2\pi i) \int_{L_j(y)} P_{jk}(t)\, P_{j,-n-1}(t) = \delta_{nk}$$

for $j, k = 0, 1, 2,..., n = 1, 2, 3,...$.

References

1. G. Faber, Über Tchebycheffsche Polynome, *Crelles J.* **150** (1920), 79–106.
2. A. Grothendieck, Sur certains espaces de fonctions holomorphes I, *J. Reine Angew. Math.* **192**, 1 (1953), 35–64.
3. J. Horvath, "Topological Vector Spaces and Distributions," Addison–Wesley, Reading, MA, 1966.
4. V. Manjarrez, Analytic function bases for multiply-connected regions, *Trans. Amer. Math. Soc.* **154** (1971).
5. W. F. Newns, On the representation of analytic functions by infinite series, *Phil. Trans. Royal Soc. London A* **245** (1953), 429–468.
6. J. L. Walsh, "Interpolation and Approximation by Rational Functions in the Complex Plane," 3rd ed., Amer. Math. Soc. Colloq. Publ., Vol. 20, Providence, RI, 1960.